

NOTES ON CARDINAL'S MATRICES

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ABSTRACT. These notes are motivated by the work of Jean-Paul Cardinal on certain symmetric matrices related to the Mertens function. He showed that certain norm bounds on his matrices implied the Riemann hypothesis. Using a different matrix norm we show an equivalence of the Riemann hypothesis to suitable norm bounds on his Mertens matrices in the new norm. We also specify a deformed version of his Mertens matrices that unconditionally satisfy a norm bound of the same strength as this Riemann hypothesis bound.

1. INTRODUCTION

In 2010 Jean-Paul Cardinal [3] introduced for each $n \geq 1$ two symmetric integer matrices \mathcal{U}_n and \mathcal{M}_n constructed using a set of “approximate divisors” of n , defined below. Each of these matrices can be obtained from the other. Both matrices are $s \times s$ matrices, where $s = s(n)$ is about $2\sqrt{n}$. Here $s(n)$ counts the number of distinct values taken by $\lfloor \frac{n}{k} \rfloor$ when k runs from 1 to n . These values comprise the approximate divisors, and we label them $k_1 = 1, k_2, \dots, k_s = n$ in increasing order.

We start with defining Cardinal’s basic matrix \mathcal{U}_n attached to an integer n . The (i, j) -th entry of \mathcal{U}_n is $\lfloor \frac{n}{k_i k_j} \rfloor$. One can show that such matrices take value 1 on the anti-diagonal, and are 0 at all entries below the anti-diagonal, consequently it has determinant ± 1 . (The anti-diagonal entries are the (i, j) -entries with $i + j = s + 1$, numbering rows and columns from 1 to s .) All entries above the anti-diagonal are positive. It follows that this matrix has $s^2 \approx 4n$ entries and about half of them are nonzero. The matrices \mathcal{U}_n encode information mixing together the additive and multiplicative structures of the integers in an interesting way, using also the floor function.

Cardinal defines a second matrix, \mathcal{M}_n , by the recipe

$$\mathcal{M}_n := T(\mathcal{U}_n)^{-1}T,$$

where T is a square matrix of the same size, having 1’s on and above the anti-diagonal, and value 0 at all entries strictly below the anti-diagonal. Clearly $\det(\mathcal{M}_n) = \det(\mathcal{U}_n)$. One can prove that \mathcal{M}_n has a similar pattern of entries to \mathcal{U}_n , in having all values 0 strictly below the anti-diagonal and all values 1 on the anti-diagonal, but it may now have some negative entries above the anti-diagonal. One of Cardinal’s main results is that the entries above the anti-diagonal its (i, j) -th entry is $M(\lfloor \frac{n}{k_i k_j} \rfloor)$, where $M(x) = \sum_{j \leq x} \mu(j)$ is the Mertens function of prime number theory. The entries below the

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antidiagonal are also Mertens function values, since in that case $M(\lfloor \frac{n}{k_i k_j} \rfloor) = M(0) = 0$. We might therefore name this matrix a *Mertens matrix*.

Cardinal [3, Theorem 24] proves that an upper bound $O(n^{\frac{1}{2}+\epsilon})$ on the growth of the norms of the matrices \mathcal{M}_n measured in the ℓ_2 operator norm implies the Riemann hypothesis holds. He gives numerical plots of the norms of \mathcal{M}_n for small n supporting this upper bound.

Cardinal's results are structural. He relates these matrices to finite-dimensional quotient algebras of the algebra of Dirichlet series, and proves that his quotient algebras are commutative and associative matrix algebras, which are lower triangular. His Proposition 3.5 about floor functions is important in establishing that certain linearly transformed versions of the matrices in the algebra give rise to symmetric matrices, including \mathcal{U}_n and \mathcal{M}_n .

In the 2008 French preprint version [4] of this paper Cardinal introduced additionally a deformed version $\tilde{\mathcal{U}}_n^+$ of his matrix \mathcal{U}_n , whose entries when rounded down by the floor function yield \mathcal{U}_n . He then proposed to define by the same recipe a deformed version of the Mertens matrix \mathcal{M}_n as

$$\tilde{\mathcal{M}}_n^+ := T_s(\tilde{\mathcal{U}}_n^+)^{-1}T_s.$$

He presented an argument giving a matrix norm for the perturbed matrix, asserting that it satisfied the Riemann hypothesis bound. Unfortunately his argument failed because his definition of a deformed matrix $\tilde{\mathcal{U}}_n^+$ had rank one, so was not invertible. Perhaps because of this no discussion of deformed matrices appeared in the author's final English version of his paper. Cardinal's argument, though flawed, has serious content, and we obtain below a modified result where it works.

1.1. Results. Much of this note consists of an exposition of Cardinal's results in a very slightly different notation, presenting similar numerical examples. The main new contributions are the following.

- (1) We use a different family of matrix norms, the Frobenius norms. We prove that the Riemann hypothesis is equivalent to a suitable growth bound on the Frobenius matrix norms $\|\mathcal{M}_n\|_F$ (Theorem 4.4). The choice of the family of norms possibly matters to obtain an equivalence because the dimensionality of the matrices goes to infinity as $n \rightarrow \infty$.
- (2) We introduce in Section 6.2 a natural modified definition of deformed matrix $\tilde{\mathcal{U}}_n$, that applies the floor function to "half" of Cardinal's deformed matrix, which is entrywise close to \mathcal{U}_n . We set $\tilde{\mathcal{M}}_n := T(\tilde{\mathcal{U}}_n)^{-1}T$ and prove for $\tilde{\mathcal{M}}_n$ unconditionally a Frobenius norm upper bound of the same strength as the Riemann hypothesis bound above (Corollary 6.5). This proof follows the ideas given in Cardinal's 2008 French preprint [4].

1.2. Related work. There is earlier work on integer matrices having a determinant related to the Mertens function ([1], [15], [2], [20], [21], [6]). This work concerns *Redheffer's matrix*, named after Redheffer [14]. It is an interesting question to determine if there are relations between Redheffer's matrix and Cardinal's matrix. Recent work of Cardon [5] makes a connection of the Redheffer matrix with the quantities $\lfloor \frac{n}{k} \rfloor$ appearing in Cardinal's matrix.

1.3. Matrix norms. In this paper we bound matrices using the Frobenius matrix norm

$$||M||_F^2 := \sum_{i=1}^n \sum_{j=1}^n |M_{ij}|^2.$$

In Cardinal's paper [3] the matrix norm $||M||$ on an $n \times n$ complex matrix is taken to be the l_2 operator norm

$$||M|| := \max_{v \in \mathbb{C}^n, v \neq 0} \frac{||Mv||^2}{||v||^2},$$

where $||v||^2 = \sum_{i=1}^n |v_i|^2$. For symmetric real matrices M this norm bound coincides with the spectral radius of the matrix.

2. CARDINAL'S ALGEBRA OF APPROXIMATE DIVISORS

Cardinal's paper is concerned with matrices encoding properties of "approximate divisors" of n . The surprising property they have is of forming a commutative subalgebra \mathcal{A}_n of lower triangular matrices, of rank $s = s(n)$.

These matrices encode information mixing together the additive and multiplicative structures of the integers in an interesting way.

2.1. Approximate Divisors. First, Cardinal introduces "approximate divisors" of n (our terminology). The number of such divisors is on the order of twice the square root of n .

Definition 2.1. Let \mathcal{S}_n be the set of distinct integers of the form $\{\lfloor \frac{n}{k} \rfloor : 1 \leq k \leq n\}$. Let

$$s = s(n) := \#(\mathcal{S}_n),$$

so that $s(n) = 2\lfloor \sqrt{n} \rfloor$, or $2\lfloor \sqrt{n} \rfloor - 1$.

The set $\mathcal{S}_n = \mathcal{S}_n^- \cup \mathcal{S}_n^+$ consists of all integers in

$$\mathcal{S}_n^- := \{j : 1 \leq j \leq \lfloor \sqrt{n} \rfloor\},$$

together with the complementary set

$$\mathcal{S}_n^+ := \{\lfloor \frac{n}{j} \rfloor : 1 \leq j \leq \lfloor \sqrt{n} \rfloor\}.$$

These sets are disjoint if $m(m+1) \leq n < (m+1)^2$ and they have exactly one element in common, namely $m = \lfloor \sqrt{n} \rfloor$, if $m^2 \leq n < m(m+1)$. He shows [3, Prop. 4]:

Lemma 2.2. (Cardinal) Number the elements of \mathcal{S} in increasing order as k_i , $1 \leq i \leq s$, so $k_1 = 1, k_s = n$. The map $\hat{\cdot} : \mathcal{S} \rightarrow \mathcal{S}$ defined by

$$\hat{k}_i := \lfloor \frac{n}{k_i} \rfloor = k_{s+1-i},$$

is an involution exchanging \mathcal{S}_n^+ and \mathcal{S}_n^- .

Example 2.3. For $n = 16$ the approximate divisors are $\{1, 2, 3, 4, 5, 8, 16\}$. The involution acts $\hat{1} = 16$, $\hat{16} = 1$, and $\hat{2} = 8$, $\hat{8} = 2$, and $\hat{3} = 5$, $\hat{5} = 3$ and $\hat{4} = 4$.

2.2. Cardinal's Multiplication Algebra \mathcal{A}_n . Cardinal associates to an integer n a commutative, associative algebra \mathcal{A}_n of lower triangular matrices of dimension s inside the $s \times s$ matrices, for which he gives generators. The generators give the effect of "multiplication by a fixed divisor" on the this algebra.

Theorem 2.4. (Cardinal) *The algebra \mathcal{A}_n has rank s and is spanned by the $s \times s$ matrices $\{\rho_n(k); k \in \mathcal{S}_n\}$, where k runs over the set of approximate divisors of n , which are all integers of the form $\lfloor \frac{n}{k} \rfloor$. Each matrix $\rho_n(k)$ is a lower triangular matrix giving the effect of multiplication by k on a basis of approximate divisors, arranged in increasing order. The matrix $\rho(1)$ is the identity matrix, and all other $\rho_n(k)$ are lower triangular nilpotent. This algebra is commutative.*

The commutativity property is a consequence of the identities in Proposition 3.5.

Example 2.5. For $n = 16$ the multiplication by k matrices $\rho(k)$ for $k = 2, 3, 4, 5$ are given below (omitted entries are 0). All these matrices are nilpotent.

$$\rho_{16}(2) = \begin{array}{c|cccccccc} \rho_{16}(2) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & \\ 2 & 1 & & & & & & \\ 3 & & & & & & & \\ 4 & & 1 & & & & & \\ 5 & & & & & & & \\ 8 & & & 1 & 1 & & & \\ 16 & & & & & 1 & 1 & \end{array}$$

$$\rho_{16}(3) = \begin{array}{c|cccccccc} \rho_{16}(3) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & \\ 2 & & & & & & & \\ 3 & 1 & & & & & & \\ 4 & & & & & & & \\ 5 & & & & & & & \\ 8 & & 1 & & & & & \\ 16 & & & 1 & 1 & 1 & & \end{array}$$

$$\rho_{16}(4) = \begin{array}{c|cccccccc} \rho_{16}(4) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ 4 & 1 & & & & & & \\ 5 & & & & & & & \\ 8 & & 1 & & & & & \\ 16 & & & 1 & 1 & & & \end{array}$$

$$\rho_{16}(5) = \begin{array}{c|cccccccc} \rho_{16}(5) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ 4 & & & & & & & \\ 5 & & 1 & & & & & \\ 8 & & & & & & & \\ 16 & & & 1 & 1 & & & \end{array}$$

Remark 2.6. The algebra \mathcal{A}_n is not semi-simple. In fact all its generators, aside from the identity element, are nilpotent. It has dimension exactly s , equal to its size. (Commutative subalgebras of matrix algebras over \mathbb{C} can have larger dimension than their size s for $s \geq 6$. The maximal dimension is $\lfloor \frac{1}{4}s^2 \rfloor + 1$, a result found by I. Schur [?]. A simple proof was given by Mirzakhani [12]).

2.3. Dirichlet Series and Cardinal's Multiplication Algebra \mathcal{A}_n . Cardinal showed the algebra \mathcal{A}_n to be a homomorphic image of the algebra of formal Dirichlet series. Let $\mathcal{D}_{\mathbb{Z}}$ denote the \mathbb{Z} -algebra of all (formal) Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with integer coefficients $a_n \in \mathbb{Z}$. Here addition on Dirichlet series coefficients is pointwise and multiplication is Dirichlet convolution, see Tenenbaum [18, Sect. 2.3]. We describe a (formal) Dirichlet series by the row vector $\mathbf{u} = (a_1, a_2, a_3, \dots)$ of its coefficients. The invertible elements in $\mathcal{D}_{\mathbb{Z}}$ are those with $a_1 = \pm 1$.

If we restrict to the subalgebra $\mathcal{D}_{\mathbb{Z}}^c$ of Dirichlet series that absolutely converge to a function $f(s)$ on some right half-plane, then viewed as such functions on a half-plane, these algebra operations correspond to pointwise addition and multiplication of the functions in their joint convergence domain.

For an integer $1 \leq j < n$, we let the *minus operator* give to each k the value k^- that is its predecessor of \mathcal{S}_n , and we artificially define $1^- := 0$. If $1 \leq k < \lfloor \sqrt{n} \rfloor$ then $k^- = k - 1$.

Cardinal's analysis implies the following result [5, Propositions 9 to 13.].

Theorem 2.7. (Cardinal) *The map $\tilde{\rho}_n : \mathcal{D}_{\mathbb{Z}} \rightarrow \mathcal{A}_n$ given by*

$$\sum_{k \geq 1} a_k k^{-s} \mapsto \sum_{m \in \mathcal{S}_n} \left(\sum_{m^- < k \leq m} a_k \right) \rho_n(m)$$

is an algebra homomorphism.

The kernels $\ker(\tilde{\rho}_n)$ of these maps seem worthy of further study, but we will not treat them here.

2.4. Matrix Image of Dirichlet series $\zeta(s)$ and $\frac{1}{\zeta(s)}$. We look at the image of the Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ and its inverse $1/\zeta(s) = \sum_{k=1}^{\infty} \mu(k)k^{-s}$ under this homomorphism.

View n as fixed. Define the vector $\mathbf{u} := \mathbf{u}_n$ to be a $1 \times s(n)$ row vector that has entries that sum the function 1 over those integers in the half-open intervals $(m^-, m]$

for $m \in \mathcal{S}_n$. The lower triangular matrix $Z_n := \rho_n(\mathbf{u}_n)$ is then a finite analogue of the Riemann zeta function. Cardinal also introduces a vector $\mu := \mu_n$ whose entries sum the Möbius function over the interval $(k^-, k]$; the corresponding inverse matrix $Z_n^{-1} := \rho_n(\mu_n)$ is then a finite analogue of the inverse of the Riemann zeta function.

Example 2.8. For $n = 16$ the vector $\mathbf{u}_{16} = (1, 1, 1, 1, 1, 3, 8)$. The matrix Z_n for $\rho_{16}(\mathbf{u})$ is (omitting 0 entries)

$$Z_{16} = \begin{array}{c|cccccccc} \rho_{16}(\mathbf{u}) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 1 & & & & & & \\ 2 & 1 & 1 & & & & & \\ 3 & 1 & 0 & 1 & & & & \\ 4 & 1 & 1 & 0 & 1 & & & \\ 5 & 1 & 0 & 0 & 0 & 1 & & \\ 8 & 3 & 2 & 1 & 1 & 0 & 1 & \\ 16 & 8 & 4 & 3 & 2 & 2 & 1 & 1 \end{array}$$

Example 2.9. For $n = 16$ the inverse (Möbius) vector is $\mu = (1, -1, -1, 0, -1, 0, 1)$. The matrix $Z_n^{-1} = \rho_{16}(\mu)$ is

$$(Z_{16})^{-1} = \begin{array}{c|cccccccc} \rho_{16}(\mu) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 1 & & & & & & \\ 2 & -1 & 1 & & & & & \\ 3 & -1 & 0 & 1 & & & & \\ 4 & 0 & -1 & 0 & 1 & & & \\ 5 & -1 & 0 & 0 & 0 & 1 & & \\ 8 & 0 & -1 & -1 & -1 & 0 & 1 & \\ 16 & 1 & -1 & -2 & -1 & -2 & -1 & 1 \end{array}$$

3. CARDINAL'S ALGEBRA OF SYMMETRIC MATRICES

3.1. Cardinal's Algebra has symmetric matrix image. A key observation of Cardinal concerns the image of the algebra \mathcal{A}_n under left multiplication by a matrix T having ones above and on its antidiagonal $a_{i,s+1-i}$, and zeros below the antidiagonal. An example of T for $s = 7$ is given in example 3.2.

Cardinal showed ([3, Proposition 20]) the following result.

Theorem 3.1. (Cardinal) *The linear map $T : \mathcal{A}_n \rightarrow \text{Mat}_{s(n) \times s(n)}$ given by $A \mapsto TA$ has image in the set of real symmetric matrices $\text{Sym}_{s(n) \times s(n)}$.*

To put this result in perspective, when T multiplies on the left a general lower triangular matrix L , the resulting matrix TL is in general *not* symmetric. This symmetry property is a special property of elements of the algebra \mathcal{A}_n . This symmetry property encodes the involution $k \mapsto \widehat{k}$, which in turn encodes the identity in Proposition 3.5.

Note that this map $A \mapsto TA$ is not an algebra homomorphism, it preserves addition but it does not respect matrix multiplication in general, i.e. usually T does not commute with all the elements of \mathcal{A}_n .

Example 3.2. For $n = 16$, we have $s(n) = 7$ and the matrix T is (omitted entries are 0). The borders giving the values of the approximate divisors are not part of the matrix.

$$T = \begin{array}{c|cccccccc} T & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 3 & 1 & 1 & 1 & 1 & 1 & & \\ 4 & 1 & 1 & 1 & 1 & & & \\ 5 & 1 & 1 & 1 & & & & \\ 8 & 1 & 1 & & & & & \\ 16 & 1 & & & & & & \end{array}$$

Example 3.3. For $n = 16$, multiplication of the matrix $\rho(2)$ by T yields a symmetric matrix $T\rho_{16}(2)$ (omitted entries are 0).

$$T\rho_{16}(2) = \begin{array}{c|cccccccc} T\rho_{16}(2) & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 2 & 1 & 1 & 1 & 1 & & & \\ 3 & 1 & 1 & & & & & \\ 4 & 1 & 1 & & & & & \\ 5 & 1 & & & & & & \\ 8 & 1 & & & & & & \\ 16 & & & & & & & \end{array}$$

3.2. Cardinal's \mathcal{U} -matrices. Cardinal now introduces an $s \times s$ symmetric matrix \mathcal{U}_n corresponding to the zeta vector \mathbf{u} :

$$\mathcal{U}_n := T_s Z_n = T_s \rho_n(\mathbf{u}).$$

The following characterization of this matrix shows that it encodes the multiplication for the approximate divisors in a manifestly symmetric form, which we stated as the definition in the introduction of the paper.

Lemma 3.4. (Cardinal) *Let $K = \text{diag}(k_1, \dots, k_s)$, where the k_j run through the elements of $\mathcal{S} = \mathcal{S}_n$ in increasing order. Thus $\widehat{k_i} = k_{s+1-i}$. Then $\mathcal{U} = \mathcal{U}_n$ is the $s \times s$ integer matrix with entries $\mathcal{U}_{i,j} = \lfloor \frac{n}{k_i k_j} \rfloor$. That is,*

$$\mathcal{U}_n = \begin{pmatrix} \lfloor n/(k_1 k_1) \rfloor & \lfloor n/(k_1 k_2) \rfloor & \cdots & \lfloor n/(k_1 k_{s-1}) \rfloor & \lfloor n/(k_1 k_s) \rfloor \\ \lfloor n/(k_2 k_1) \rfloor & \lfloor n/(k_2 k_2) \rfloor & \cdots & \lfloor n/(k_2 k_{s-1}) \rfloor & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lfloor n/(k_{s-1} k_1) \rfloor & \lfloor n/(k_{s-1} k_2) \rfloor & \cdots & 0 & 0 \\ \lfloor n/(k_s k_1) \rfloor & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The fact that the matrix in Lemma 3.4 is symmetric easily follows from the following set of identities for the floor function [3, Lemma 6].

Proposition 3.5. (Cardinal) *For all positive integers n, i, j , there holds*

$$\lfloor \frac{1}{i} \lfloor \frac{n}{j} \rfloor \rfloor = \lfloor \frac{1}{j} \lfloor \frac{n}{i} \rfloor \rfloor = \lfloor \frac{n}{ij} \rfloor.$$

These identities may be checked by a calculation. In fact that dilated floor function $\lfloor \alpha x \rfloor$ and $\lfloor \beta x \rfloor$ never commute for real (α, β) except for the discrete family $(\alpha, \beta) =$

$(\frac{1}{i}, \frac{1}{j})$ with i, j positive integers as given in Proposition 3.5, aside from three continuous families: $\alpha = \beta$, $\alpha = 0$ and $\beta = 0$, see [10].

Note that all entries on the anti-diagonal $i + j = s + 1$ of \mathcal{U}_n are 1, since

$$\mathcal{U}_{i,s+1-i} = \lfloor \frac{n}{k_i k_{s-i+1}} \rfloor = 1$$

Example 3.6. For $n = 16$ the matrix \mathcal{U}_n is (omitted entries are 0)

$$\mathcal{U}_{16} = \begin{array}{c|cccccccc} \mathcal{U}_{16} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 16 & 8 & 5 & 4 & 3 & 2 & 1 \\ 2 & 8 & 4 & 2 & 2 & 1 & 1 & \\ 3 & 5 & 2 & 1 & 1 & 1 & & \\ 4 & 4 & 2 & 1 & 1 & & & \\ 5 & 3 & 1 & 1 & & & & \\ 8 & 2 & 1 & & & & & \\ 16 & 1 & & & & & & \end{array}$$

The matrix \mathcal{U}_n is invertible, and since $\det(\mathcal{U}_n) = \pm 1$ it is also an integer matrix.

Example 3.7. For the case $n = 16$ the matrix \mathcal{U}_n^{-1} is (omitted entries are 0)

$$(\mathcal{U}_{16})^{-1} = \begin{array}{c|cccccccc} (\mathcal{U}_{16})^{-1} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & 1 \\ 2 & & & & & & 1 & -2 \\ 3 & & & & & 1 & -1 & -1 \\ 4 & & & & 1 & -1 & -1 & 1 \\ 5 & & & 1 & -1 & 0 & 0 & -1 \\ 8 & & 1 & -1 & -1 & 0 & 0 & 1 \\ 16 & 1 & -2 & -1 & 1 & -1 & 1 & 2 \end{array}$$

4. CARDINAL'S MATRIX \mathcal{M}_n AND THE MERTENS FUNCTION

Cardinal also introduces an $s \times s$ symmetric matrix \mathcal{M}_n , that corresponds to the Möbius vector μ similarly.

4.1. Cardinal's Matrix \mathcal{M}_n .

Definition 4.1. The $s \times s$ symmetric matrix \mathcal{M}_n is defined by

$$\mathcal{M} = \mathcal{M}_n := T_s \mathcal{U}_n^{-1} T_n,$$

in which T_n is an $s(n) \times s(n)$ matrix having $T_{ij} = 1$ if $i + j \leq s(n) + 1$, and 0 otherwise (i.e. if $s(n) + 2 \leq i + j \leq 2s(n)$).

Note that by definition of \mathcal{U}_n , one also has

$$\mathcal{M}_n = T_s(Z_n)^{-1} = T_s \rho(\mu).$$

Example 4.2. For $n = 16$ the matrix \mathcal{M}_n is: (omitted entries are 0)

$$\mathcal{M}_{16} = \begin{array}{c|cccccccc} \mathcal{M}_{16} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & -1 & -2 & -2 & -1 & -1 & 0 & 1 \\ 2 & -2 & -1 & 0 & 0 & 1 & 1 & \\ 3 & -2 & 0 & 1 & 1 & 1 & & \\ 4 & -1 & 0 & 1 & 1 & & & \\ 5 & -1 & 1 & 1 & & & & \\ 8 & 0 & 1 & & & & & \\ 16 & 1 & & & & & & \end{array}$$

A main result of Cardinal is that the entries of this matrix are expressible using the *Mertens function*

$$M(x) := \sum_{1 \leq j \leq \lfloor x \rfloor} \mu(j).$$

Here we set $M(x) = 0$ for $0 \leq x < 1$.

Theorem 4.3. (Cardinal) *The entries of \mathcal{M}_n are exactly the Mertens function evaluation of the entries of \mathcal{U}_n , i.e.*

$$(\mathcal{M}_n)_{ij} = M(\mathcal{U}_{ij}), \text{ for } i, j \in \mathcal{S}_n,$$

where $M(x)$ is the *Mertens function*. That is

$$\mathcal{M}_n = \begin{pmatrix} M(\lfloor n/(k_1 k_1) \rfloor) & M(\lfloor n/(k_1 k_2) \rfloor) & \cdots & M(\lfloor n/(k_1 k_{s-1}) \rfloor) & M(\lfloor n/(k_1 k_s) \rfloor) \\ M(\lfloor n/(k_2 k_1) \rfloor) & M(\lfloor n/(k_2 k_2) \rfloor) & \cdots & M(\lfloor n/(k_2 k_{s-1}) \rfloor) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M(\lfloor n/(k_{s-1} k_1) \rfloor) & M(\lfloor n/(k_{s-1} k_2) \rfloor) & \cdots & 0 & 0 \\ M(\lfloor n/(k_s k_1) \rfloor) & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Here again one has that all the anti-diagonal entries $i + j = s + 1$ are given by

$$\mathcal{M}_{i, s+1-i} = M(\lfloor \frac{n}{k_i k_{s-i+1}} \rfloor) = M(1) = 1.$$

4.2. Riemann Hypothesis equivalence in terms of \mathcal{M}_n . The Riemann hypothesis is nicely encoded in terms of a suitable bound on the norms of these matrices \mathcal{M}_n .

Theorem 4.4. *The following properties are equivalent.*

(i) *For each $\epsilon > 0$ there holds the estimate*

$$\|\mathcal{M}_n\|_F = O_\epsilon(n^{\frac{1}{2}+\epsilon}).$$

where $\|\mathcal{M}\|_F^2 = \sum_{i=1}^s \sum_{j=1}^s \mathcal{M}_{ij}^2$ is the *Frobenius matrix norm*.

(ii) *The Riemann hypothesis holds.*

Proof. (i) \Rightarrow (ii). The hypothesis (i) yields the upper bound

$$M(n)^2 = |\mathcal{M}_{11}|^2 \leq \|\mathcal{M}_n\|_F^2 = O_\epsilon(n^{1+2\epsilon}).$$

This upper bound on the the Mertens function is well-known to imply the Riemann hypothesis [19, Theorem 14.25(C)].

(ii) \Rightarrow (i). The Mertens function RH bound implies that

$$\begin{aligned}
\|\mathcal{M}_n\|_F^2 &= \sum_{i=1}^s \sum_{j=1}^s \mathcal{M}_{ij}^2 \\
&\leq \sum_{i=1}^s \sum_{j=1}^s M\left(\frac{n}{k_i k_j}\right)^2 \\
&= O_\epsilon \left(\sum_{i=1}^s \sum_{j=1}^s \left(\frac{n}{k_i k_j}\right)^{1+\epsilon} \right) \\
&= O_\epsilon \left(n^{1+\epsilon} \left(\sum_{i=1}^\infty \left(\frac{1}{k_i}\right)^{1+\epsilon} \right) \left(\sum_{j=1}^\infty \left(\frac{1}{k_j}\right)^{1+\epsilon} \right) \right) \\
&= O_\epsilon (n^{1+\epsilon}).
\end{aligned}$$

Here we used the fact that the Mertens function is constant on unit intervals to remove the greatest integer function. The important thing is that the implied constant in the O -symbol does not depend on $s = s(n)$. \square

Cardinal observes empirically that the function $\|\mathcal{M}_n\|$ (using the l_2 operator norm) seems to be a much smoother function than the Mertens function $M(n)$, i.e. it empirically has smaller fluctuations.

Remark 4.5. The Frobenius norm $\|\mathcal{M}_n\|_F$ must be at least as large as its $(1, 1)$ -th entry. Hence it must see fluctuations as large as those of $M(n)$ in the upwards direction.

The true order of growth of the Mertens function, assuming the truth of RH, is not known at present. Soundararajan [17] shows that the RH implies the upper bound

$$|M(n)| = O\left(\sqrt{n} \exp((\log x)^{1/2} (\log \log x)^{14})\right)$$

His paper suggests that assuming conjectured bounds on the maximal size of L-functions (made by Farmer, Gonek and Hughes) one might be able to derive the stronger bound

$$|M(n)| =_? O\left(\sqrt{n} \exp\left(C(\log \log x)^3\right)\right).$$

Heuristics suggest that the upper bound is much smaller. Nathan Ng [Ng04, Theorem 1] shows, assuming unproved hypotheses, that

$$|M(n)| \leq \sqrt{n} (\log n)^{\frac{3}{2}}.$$

He notes that S. Gonek has conjectured that the best possible maximal order of magnitude of $M(n)$ will be of shape

$$|M(x)| \leq B\sqrt{n} (\log \log \log n)^{5/4},$$

where $B > 0$ is a constant. In the other direction it is known that the GUE hypothesis implies that

$$\limsup_{n \rightarrow \infty} \frac{|M(n)|}{\sqrt{n}} = +\infty.$$

Kaczorowski [7] has proved a complementary result showing that for a certain analogue $M^*(n)$ of $M(n)$ the fluctuations of $M^*(n)$ are at least of size $\sqrt{n} \log \log \log n$.

5. DEFORMED VERSION OF CARDINAL MATRIX \mathcal{U}_n

We define and study certain deformed matrices $\tilde{\mathcal{U}}_n$, of similar design to \mathcal{U}_n . All entries of $\tilde{\mathcal{U}}_n$ are greater than or equal to the corresponding entry of \mathcal{U}_n , and any entry increase is less than one. Thus \mathcal{U}_n is recoverable from $\tilde{\mathcal{U}}_n$ by applying the floor function to each of its entries. These are modification of the deformation proposed in [4].

5.1. Deformed Matrix $\tilde{\mathcal{U}}_n$. We now define and study certain deformed matrices $\tilde{\mathcal{U}}_n$, of similar design to \mathcal{U}_n . All entries of $\tilde{\mathcal{U}}_n$ are greater than or equal to the corresponding entry of \mathcal{U}_n , and any increase is less than one. Thus \mathcal{U}_n is recoverable from $\tilde{\mathcal{U}}_n$ by applying the floor function to each of its entries.

Definition 5.1. For $n \in \mathbb{N}$, let $T = T_s$ be the $s \times s$ matrix with $s = s(n) = \#(\mathcal{S}_n)$, which has ones on and above the antidiagonal, and zeros elsewhere. That is, $T_s = (t_{ij})$, where $t_{ij} = 1$ if $i + j \leq s + 1$, and 0 else.

$$T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Its inverse T^{-1} is the matrix with 1's on the antidiagonal and (-1) 's just below the antidiagonal. That is:

$$T^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & -1 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Definition 5.2. For $n \in \mathbb{N}$, define the row vector

$$\mathbf{d} = (d_1, d_2, \dots, d_s) := \sqrt{n}/k_1, \sqrt{n}/k_2, \dots, \sqrt{n}/k_s,$$

where k_j runs through $\mathcal{S} = \mathcal{S}_n$ in increasing order. Let D be the diagonal matrix $D = \text{diag}(d_1, \dots, d_s)$, and then set

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_n := DTD.$$

Then set

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_n := T\tilde{\mathcal{U}}^{-1}T.$$

We note an equivalent definition of $\tilde{\mathcal{U}}_n$.

Proposition 5.3. *The matrix $\tilde{\mathcal{U}}_n$ is equal to*

$$\tilde{\mathcal{U}}_n = \begin{pmatrix} n/(k_1 k_1) & n/(k_1 k_2) & \cdots & n/(k_1 k_{s-1}) & n/(k_1 k_s) \\ n/(k_2 k_1) & n/(k_2 k_2) & \cdots & n/(k_2 k_{s-1}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n/(k_{s-1} k_1) & n/(k_{s-1} k_2) & \cdots & 0 & 0 \\ n/(k_s k_1) & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where k_i runs through \mathcal{S} in increasing order. That is, $\tilde{\mathcal{U}} = (\tilde{u}_{ij})$, where

$$\tilde{u}_{ij} = \begin{cases} n/(k_i k_j) & : i + j \leq s + 1 \\ 0 & i + j \geq s + 2 \end{cases}$$

Note the simple relationship: \mathcal{U}_n is obtained from $\tilde{\mathcal{U}}_n$ by taking the greatest integer part of each entry of $\tilde{\mathcal{U}}_n$.

Proof. As defined, $\tilde{\mathcal{U}} = DTD$, and we have

$$\begin{aligned} DTD &= \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{s-1} & 0 \\ 0 & 0 & \cdots & 0 & d_s \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{s-1} & 0 \\ 0 & 0 & \cdots & 0 & d_s \end{pmatrix} \\ &= \begin{pmatrix} d_1 & d_1 & \cdots & d_1 & d_1 \\ d_2 & d_2 & \cdots & d_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{s-1} & d_{s-1} & \cdots & 0 & 0 \\ d_s & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{s-1} & 0 \\ 0 & 0 & \cdots & 0 & d_s \end{pmatrix} \\ &= \begin{pmatrix} d_1 d_1 & d_1 d_2 & \cdots & d_1 d_{s-1} & d_1 d_s \\ d_2 d_1 & d_2 d_2 & \cdots & d_2 d_{s-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{s-1} d_1 & d_{s-1} d_2 & \cdots & 0 & 0 \\ d_s d_1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \end{aligned}$$

The proposition follows from noting that the entries of this final matrix are equal to the \tilde{u}_{ij} defined in the proposition statement. \square

Example 5.4. For the case $n = 16$ the matrix $\tilde{\mathcal{U}}_n$ is (omitted entries are 0)

$$\tilde{\mathcal{U}}_{16} = \begin{array}{c|ccccccc} \tilde{\mathcal{U}}_{16} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 16 & 8 & 16/3 & 4 & 16/5 & 2 & 1 \\ 2 & 8 & 4 & 8/3 & 2 & 8/5 & 1 & \\ 3 & 16/3 & 8/3 & 16/9 & 4/3 & 16/15 & & \\ 4 & 4 & 2 & 4/3 & 1 & & & \\ 5 & 16/5 & 8/5 & 16/15 & & & & \\ 8 & 2 & 1 & & & & & \\ 16 & 1 & & & & & & \end{array}$$

The matrix $\tilde{\mathcal{U}}_n$ can be viewed as a one-sided perturbation of \mathcal{U}_n in which each entry is larger by some amount between 0 and 1.

We next define a matrix $\tilde{\mathcal{U}}^+ = \hat{\mathcal{U}}_n^+ := \mathbf{d}^T \mathbf{d}$ (outer product), which has entries

$$\hat{\mathcal{U}}_{ij}^+ = n/(k_i k_j), \quad 1 \leq i, j \leq s.$$

In the French preprint [4] this was Cardinal's formal definition of a matrix which he denoted $\tilde{\mathcal{U}}$. This matrix has rank one, in consequence Cardinal's formal definition of $\tilde{\mathcal{M}}$, which uses $\tilde{\mathcal{U}}^{-1}$, becomes undefined.

Example 5.5. For the case $n = 16$ the outer product matrix $\tilde{\mathcal{U}}_{16}^+ = \mathbf{d}^T \mathbf{d}$ is:

$$\tilde{\mathcal{U}}_{16}^+ =$$

$\tilde{\mathcal{U}}_{16}^+$	1	2	3	4	5	8	16
1	16	8	16/3	4	16/5	2	1
2	8	4	8/3	2	8/5	1	1/2
3	16/3	8/3	16/9	4/3	16/15	2/3	1/3
4	4	2	4/3	1	4/5	1/2	1/4
5	16/5	8/5	16/15	4/5	16/25	2/5	1/5
8	2	1	2/3	1/2	2/5	1/4	1/8
16	1	1/2	1/3	1/4	1/5	1/8	1/16

This matrix has rank one, and agrees with $\tilde{\mathcal{U}}_{16}$ on and above the antidiagonal. The difference matrix is:

$$\tilde{\mathcal{U}}_{16}^+ - \tilde{\mathcal{U}}_{16} =$$

$\tilde{\mathcal{U}}_{16}^+ - \tilde{\mathcal{U}}_{16}$	1	2	3	4	5	8	16
1							0
2						0	1/2
3					0	2/3	1/3
4				0	4/5	1/2	1/4
5			0	4/5	16/25	2/5	1/5
8		0	2/3	1/2	2/5	1/4	1/8
16	0	1/2	1/3	1/4	1/5	1/8	1/16

The matrix $\tilde{\mathcal{U}}_n$ can be viewed as a one-sided perturbation of \mathcal{U}_n in which each entry is larger by some amount between 0 and 1.

Remark 5.6. (1) On can recover the Cardinal matrix \mathcal{U}_n from either of the matrices $\tilde{\mathcal{U}}_n^+$ and $\tilde{\mathcal{U}}_n$ by applying the floor function to each entry. Here $\tilde{\mathcal{U}}_n$ is also obtained by applying the floor function to some entries of $\tilde{\mathcal{U}}_n^+$, namely those entries of $\tilde{\mathcal{U}}_n^+$ that are strictly below the antidiagonal (the lower right corner), and whose effect is to zero out all of these entries. As a consequence $\tilde{\mathcal{U}}$ is an invertible matrix, having determinant the product of the entries on the anti-diagonal, times $(-1)^{\frac{n(n-1)}{2}}$. Furthermore all entries on the antidiagonal are ≥ 1 , so that $|\det(\tilde{\mathcal{U}})| \geq 1$.

(2) The structure of \mathcal{U}_n^{-1} is similar in shape to that of T^{-1} . Namely it is supported on the antidiagonal and the parallel antidiagonal under it, see the next example.

Example 5.7. For the case $n = 16$ the matrix $\tilde{\mathcal{U}}_n^{-1}$ is (omitted entries are 0)

$$\tilde{\mathcal{U}}_{16} = \begin{array}{c|cccccccc} \tilde{\mathcal{U}}_{16} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & & & & & & & 1 \\ 2 & & & & & & 1 & -1/2 \\ 3 & & & & & 15/16 & -3/2 & \\ 4 & & & & 1 & -5/4 & & \\ 5 & & & 15/16 & -5/4 & & & \\ 8 & & 1 & -3/2 & & & & \\ 16 & 1 & -1/2 & & & & & \end{array}$$

6. DEFORMED MATRIX $\widetilde{\mathcal{M}}_n$

In this section we show the proof idea of Cardinal on the deformed matrix given in [4] can be adjusted to work using the modified definition of $\tilde{\mathcal{U}}_n$ given above. The argument given follows [4] closely, but uses the Frobenius norm instead of the ℓ_2 operator norm.

6.1. Deformed matrix $\widetilde{\mathcal{M}}_n$. We define the *deformed Cardinal Mertens matrix* $\widetilde{\mathcal{M}}_n$ by following the same recipe for \mathcal{M} , namely

$$\widetilde{\mathcal{M}}_n := T(\tilde{\mathcal{U}}_n)^{-1}.$$

Proposition 6.1. *For the nonnegative symmetric $s \times s$ real matrices $T_n, \mathcal{U}_n, \tilde{\mathcal{U}}_n$ the following hold.*

- (1) $T \leq_P \mathcal{U} \leq_P \tilde{\mathcal{U}}$, where \leq_P represents the inequality of corresponding entries.
- (2) For the values of \mathcal{U} in the upper-left corner of the matrix, we have $\mathcal{U} \simeq \tilde{\mathcal{U}}$, in the sense that for $i + j \leq s + 1$,

$$0 \leq \tilde{\mathcal{U}}_{ij} - \mathcal{U}_{ij} \leq 1 = T_{ij} \leq \mathcal{U}_{ij}.$$

- (3) For the values of \mathcal{U} on the antidiagonal $i + j = s + 1$, we have $\mathcal{U}_{ij} = T_{ij} = 1$.

Proof. The properties (1)-(3) follow by inspection. □

6.2. Bound for norm of deformed matrix $\|\widetilde{\mathcal{M}}_n\|_F$. Our main observation is that the norm of $\|\widetilde{\mathcal{M}}_n\|_F$ can be unconditionally estimated, and it satisfies the Riemann hypothesis bound.

To estimate the growth of $\|\widetilde{\mathcal{M}}_n\|_F$, we first need a bound for $\|T_n\|_F$.

Proposition 6.2. *The matrix $T = T_{s(n)}$ satisfies the Frobenius norm bound*

$$\|T_n\|_F = O(\sqrt{n}),$$

Proof. For a $s \times s$ matrix A , let $\max |A| = \max |a_{ij}|$. We use the general bound

$$\|A\|_F^2 = \sum_{i=1}^s \sum_{j=1}^s |A_{ij}|^2 \leq s^2 (\max |A|)^2,$$

which gives

$$\|A\|_F \leq s \max |A|.$$

Here, applied to $A = T_n$, we have $\max |A| = 1$, and $s = \#S \sim 2\sqrt{n}$. \square

Remark 6.3. It is clear that $\|T_n\|_F \geq \sqrt{n}$, since T_n has $\sim \frac{1}{2}(2\sqrt{n})^2$ entries equal to 1. In terms of the l_2 -norm, it is also easy to see that $\liminf \|T_n\|/\sqrt{n} \geq 1$. Indeed, let us consider the column vector w of size $\#S$ made up of 1s, and denote its transpose by w' . Because of the fact that $\#S \simeq 2\sqrt{n}$ we have $\|w\|^2 \simeq 2\sqrt{n}$ and also $w'Tw \simeq 2n$. The spectral radius of the symmetric matrix T_n , which is also the l_2 -norm of T_n , is thus greater than $\frac{w'Tw}{\|w\|^2} \simeq \sqrt{n}$.

Now we can bound the size of the perturbed inverse matrix $\widetilde{\mathcal{M}}_n$ elementwise, as follows.

Lemma 6.4. *The deformed matrix $\widetilde{\mathcal{M}}_n$ has elements bounded by*

$$\max_{i,j} |(\widetilde{\mathcal{M}}_n)_{i,j}| = O(\log n).$$

Proof. As $\widetilde{\mathcal{U}}_n$ is defined to be DTD , we have $(\widetilde{\mathcal{U}}_n)^{-1} = D^{-1}T^{-1}D^{-1}$. We now calculate T^{-1} , D^{-1} , and $(\widetilde{\mathcal{U}}_n)^{-1}$.

- (1) T^{-1} is the matrix with 1's on the antidiagonal and (-1) 's just below the antidiagonal. That is:

$$T^{-1} = \begin{pmatrix} & & & & 1 \\ & & & 1 & -1 \\ & & 1 & -1 & \\ & & & \ddots & \\ & 1 & -1 & & \\ 1 & -1 & & & \end{pmatrix}.$$

- (2) $D^{-1} = \text{diag}(1/\sqrt{n}, \dots, k/\sqrt{n}, \dots, n/\sqrt{n})$, where k runs through \mathcal{S} .

- (3) $(\widetilde{\mathcal{U}}_n)^{-1}$ has the same shape as T^{-1} in that the only nonzero entries are on and just below the antidiagonal.

Traversing the antidiagonal of $(\widetilde{\mathcal{U}}_n)^{-1}$ from the bottom left to the top right, we get the terms $\frac{k\bar{k}}{n}$, where k runs through \mathcal{S} . As $\bar{k} = [n/k]$, it follows that each one of these terms lies between 0 and 1.

Traversing just under the antidiagonal, we have the terms $-\frac{\bar{k}k^+}{n}$, where k runs through $\mathcal{S} \setminus \{n\}$ and k^+ designates the successor of k in \mathcal{S} . As we can bound $\frac{\bar{k}k^+}{n}$ by $\frac{(n/k)k^+}{n} = \frac{k^+}{k} \leq C$ for some constant C (as shown in point 3 of Proposition 2.4 of [4], we can take $C = 4 + 2\sqrt{2}$). Thus each of these terms thus lies between $-C$ and 0.

Now, we turn to $\widetilde{\mathcal{M}}_n = T(\mathcal{U}_n)^{-1}T$. Upon multiplying a general $s \times s$ matrix $A = (a_{ij})$ on both sides by T , we see that the resulting matrix takes the form $TAT = C = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{s+1-i} \sum_{\ell=1}^{s+1-j} a_{k\ell}.$$

Thus, we see that obtaining the (i, j) -th term of $\widetilde{\mathcal{M}}_n = T(\widetilde{\mathcal{U}}_n)^{-1}T$ consists of summing all the terms within the $(s+1-i) \times (s+1-j)$ submatrix starting located at the top left of $(\widetilde{\mathcal{U}}_n)^{-1}$. Considering the form of $(\widetilde{\mathcal{U}}_n)^{-1}$ – the only nonzero entries are on and just below the antidiagonal – and the fact that the matrix is symmetric, we conclude that each coefficient of $\widetilde{\mathcal{M}}_n$ is the sum of at most:

- two sums of terms along the antidiagonal, each of the form (the two sums are the same as the matrix is symmetric) $\frac{1}{n} \sum_{k \in \mathcal{S}, i \leq k \leq j} \bar{k}(k - k^+)$, where i and j are fixed in \mathcal{S} , and $j < \sqrt{n}$
- and a sum of at most three terms at the center of the matrix, each of which is between $-C$ and 1.

Now, since we have $k < \sqrt{n}$ in the above sums, we have that $k^+ = k + 1$ and so each of the two sums mentioned above simplifies to $-\frac{1}{n} \left(\sum_{k \in \mathcal{S}, i \leq k \leq j} \bar{k} \right)$. Now, we have the bound

$$\sum_{k \in \mathcal{S}, i \leq k \leq j} \bar{k} \leq \sum_{1 \leq k \leq \sqrt{n}} \bar{k} = \sum_{1 \leq k \leq \sqrt{n}} [n/k] \leq \sum_{1 \leq k \leq \sqrt{n}} n/k \sim n \log \sqrt{n},$$

and so we obtain

$$\max |\widetilde{\mathcal{M}}_n| = O(\log n).$$

□

Lemma 6.4 immediately yields a bound for the Frobenius matrix norm $\|\widetilde{\mathcal{M}}_n\|_F$.

Corollary 6.5. *The Frobenius norm of $\widetilde{\mathcal{M}}_n$ satisfies*

$$\|\widetilde{\mathcal{M}}_n\|_F = O(\sqrt{n} \log n).$$

Proof. Use Lemma 6.4 along with the inequality used in Proposition 6.1, which is $\|A\|_F \leq s \max |A|$. □

Remark 6.6. The Frobenius norm upper bound $O(\sqrt{n} \log n)$ for $\widetilde{\mathcal{M}}_n$ is better than the known Ω -bounds on the fluctuations of the Mertens's function, assuming RH. See Remark 4.5.

Remark 6.7. Cardinal [3] reports that numerical experiments with the l_2 -operator norm suggest that the ratio $\|\widetilde{\mathcal{M}}_n\|/(\sqrt{n} \log n)$ seems to tend to 0. He also remarks it is easy to prove that $\max |\widetilde{\mathcal{M}}_n|/\log n$ has a strictly positive limit (a result more precise than Lemma 6.4). It appears that the inequality $\|A\| \leq s \max |A|$ gives away too much to deduce such a result.

7. REMARKS ON BOUNDING CARDINAL'S MATRIX $||\mathcal{M}_n||_F$

We would like to obtain upper bounds for $||\mathcal{M}_n||_F$, perhaps viewing \mathcal{M}_n as a perturbation of $\tilde{\mathcal{M}}_n$. One approach is first study its inverse \mathcal{U}_n compared to $\tilde{\mathcal{U}}_n$. The basic quantity controlling the size of $||\mathcal{M}_n||_F$ will be the size of the *smallest* eigenvalue of \mathcal{U}_n . This is because \mathcal{M}_n is a fixed linear of the matrix $T(\mathcal{U}_n)^{-1}$ and the largest norm eigenvalue of the symmetric matrix $(\mathcal{U}_n)^{-1}$ is the reciprocal of the smallest norm (nonzero) eigenvalue of \mathcal{U}_n . Note that $\det(\mathcal{U}_n) = 1$ and $\det(\tilde{\mathcal{U}}_n) \geq 1$.

7.1. Positivity property. Each of \mathcal{U}_n and $\tilde{\mathcal{U}}_n$ and, especially, their difference matrix

$$E_n := \tilde{\mathcal{U}}_n - \mathcal{U}_n$$

are *nonnegative* symmetric matrices. Perhaps the nonnegativity constraint can be put to some use. Note that the perturbation E_n has $\det(E_n) = 0$.

Example 7.1. For the case $n = 16$ the matrix E_n is (omitted entries are 0)

$$E_{16} = \begin{array}{c|cccccccc} E_{16} & 1 & 2 & 3 & 4 & 5 & 8 & 16 \\ \hline 1 & 0 & 0 & 1/3 & 0 & 1/5 & 0 & 0 \\ 2 & 0 & 0 & 2/3 & 0 & 3/5 & 0 & \\ 3 & 1/3 & 2/3 & 7/9 & 1/3 & 1/15 & & \\ 4 & 0 & 0 & 1/3 & 0 & & & \\ 5 & 1/5 & 3/5 & 1/15 & & & & \\ 8 & 0 & 0 & & & & & \\ 16 & 0 & & & & & & \end{array}$$

Next we note that Cardinal's original deformed matrix $\tilde{\mathcal{U}}_n^+$ is given by the outer product

$$\tilde{\mathcal{U}}_n^+ := [d_1, d_2, \dots, d_s]^T [d_1 d_2, \dots, d_s] = [d_i d_j]_{1 \leq i, j \leq s}$$

which shows it is a rank one matrix, hence non-invertible. It satisfies

$$\tilde{\mathcal{U}}_n^+ \geq_N \tilde{\mathcal{U}}_n \geq_N \mathcal{U}_n.$$

where $A \geq_N B$ means $A - B$ is a nonnegative (symmetric) matrix. It follows that the larger difference matrix

$$\tilde{E}_n := \tilde{\mathcal{U}}_n^+ - \mathcal{U}_n$$

exhibits a sufficiently large nonnegative perturbation to reach a very singular matrix $\tilde{\mathcal{U}}_n^+$ from \mathcal{U}_n .

Example 7.2. For the case $n = 16$ the larger difference matrix \tilde{E}_n is

$$\tilde{E}_{16} =$$

\tilde{E}_{16}	1	2	3	4	5	8	16
1	0	0	1/3	0	1/5	0	0
2	0	0	2/3	0	3/5	0	1/2
3	1/3	2/3	7/9	1/3	1/15	2/3	1/3
4	0	0	1/3	0	4/5	1/2	1/4
5	1/5	3/5	1/15	4/5	16/25	2/5	1/5
8	0	0	2/3	1/2	2/5	1/4	1/8
16	0	1/2	1/3	1/4	1/5	1/8	1/16

This suggests that it will be hard to bound the smallest eigenvalue of \mathcal{U}_n by a general matrix inequality.

It may also be useful to study the difference matrix

$$E_n^+ := \tilde{\mathcal{U}}_n^+ - \tilde{\mathcal{U}}_n.$$

Here one has

$$\tilde{E}_n = E_n^+ + E_n.$$

Example 7.3. For the case $n = 16$ the difference matrix E_n^+ is (omitted entries are 0)

$$E_{16}^+ =$$

E_{16}^+	1	2	3	4	5	8	16
1							0
2						0	1/2
3					0	2/3	1/3
4				0	4/5	1/2	1/4
5			0	4/5	16/25	2/5	1/5
8		0	2/3	1/2	2/5	1/4	1/8
16	0	1/2	1/3	1/4	1/5	1/8	1/16

We can also define an upper triangular matrix \tilde{Z}_n by

$$\tilde{\mathcal{U}}_n = T_n \tilde{Z}_n$$

The matrix \tilde{Z}_n is lower triangular, and we can compare it with Z_n . In comparing these two matrices, in the range $1 \leq j \leq i \leq s$ we have

$$-1 < (\tilde{Z}_n - Z_n)_{i,j} < 1,$$

a result which follows from the structure of T_n^{-1} .

Example 7.4. For $n = 16$ the matrix \tilde{Z}_n is (omitted entries are 0)

$$\tilde{Z}_{16} =$$

\tilde{Z}_{16}	1	2	3	4	5	8	16
1	1						
2	1	1					
3	6/5	3/5	16/15				
4	4/5	2/5	4/15	1			
5	4/3	2/3	4/9	1/3	16/15		
8	8/3	4/3	8/9	2/3	8/15	1	
16	8	4	8/3	2	8/5	1	1

Example 7.5. For $n = 16$ the matrix $W_n := \tilde{Z}_n - Z_n$ is (omitted entries are 0)

$$W_{16} := \tilde{Z}_{16} - Z_{16} =$$

W_{16}	1	2	3	4	5	8	16
1	0						
2	0	0					
3	1/5	3/5	1/15				
4	-1/5	-3/5	4/15	0			
5	1/3	2/3	4/9	1/3	1/15		
8	-1/3	-2/3	-1/9	-1/3	8/15	0	
16	0	0	-1/3	0	-2/5	0	0

As remarked above, all entries of W_n lie strictly between -1 and 1 . The column sums of W_n are all nonnegative, since they can be shown to coincide with the first row of $\tilde{\mathcal{U}}_n - \mathcal{U}_n$.

7.2. Continuous deformations. To understand the structure of eigenvalues and eigenvectors of Cardinal's matrices \mathcal{U}_n one might try deforming the Cardinal matrix \mathcal{U}_n in other fashions, along a smooth path, in which it remains skew upper triangular, and symmetric, with numbers close to 1 on the diagonal. These are homotopy paths and it might be interesting to use $T = T_{s(n)}$ as the base point.

One can get to the matrix \mathcal{U}_n starting from the matrix T by making non-negative increasing changes in entries above the anti-diagonal, leaving zeros below the diagonal, and only allowing such deformations with all deformed matrices remaining symmetric. This preserves the reality of the eigenvalues during the deformation, which vary continuously. The eigenvalues of T are all equal to 1 and -1 , as equal as possible in number. They may mismatch in number by at most 1, noting that the trace of T is 0 or 1 depending on whether s is even or odd.

Under a continuous symmetric deformation, no eigenvalue can change sign, because a symmetric matrix has real eigenvalues and eigenvalue 0 cannot occur because the anti-diagonal elements remain positive, so the sign of the determinant stays fixed.

We note that there are about s different values of n giving Cardinal matrices \mathcal{U}_n having the same size s , because $s(n)$ jumps by 1 only at values $n = k(k+1)$ or k^2 . The Cardinal algebras \mathcal{A}_n of these different n are not the same in general. However the number of such matrices is much less than the total set of deformation parameters that maintain symmetry of the matrix, which is about $\frac{1}{4}s^2$. An interesting question is the structure of deformations for which there is an underlying rank s commutative algebra of lower triangular matrices (obtained by applying T^{-1} that simultaneously is being deformed along the deformation path. Perhaps requiring such an extra property would restrict the allowable set of deformations to better match the number of sample matrices.

8. CONCLUDING REMARKS

The Riemann hypothesis can also be related to the size of the smallest eigenvalue (in absolute value) of \mathcal{U}_n . Its truth requires that this eigenvalue not get too close to 0. This property might be viewed as a kind of level repulsion phenomenon. It might be useful to get more general information about the eigenvalues of \mathcal{U}_n .

- (1) How do the positive eigenvalues and (absolute values of) negative eigenvalues of \mathcal{U}_n interact. Do they interlace?

- (2) Does interlacing of eigenvalues hold when increasing n to $n + 1$, or whether they change in a simple way. Only a few entries in the matrix \mathcal{U}_n change in going from n to $n + 1$.
- (3) What happens to the eigenvalues at the special values $n = k^2$ or $k(k + 1)$ where the size of the Cardinal matrices increases by one. One can break this jump in half by building additional Cardinal matrices using the ceiling function instead of the floor function. This adds another row and column to the matrix and some interesting features emerge.

The Cardinal algebra \mathcal{A}_n is a kind of “finite-dimensional quantization” of the algebra of Dirichlet series. It is an interesting construction half way between addition and multiplication. It has added in a nice way “approximate divisors” which increase the number of divisors of n from $d(n)$ to about $2\sqrt{n}$. Recall that the number of divisors function $d(n) = O(n^\epsilon)$ for any $\epsilon > 0$. The commutativity property of the resulting algebra seems very important. The symmetry property of the matrices may be a finite-dimensional vestige of the symmetry under $s \rightarrow 1 - s$ given in the functional equation of the Riemann zeta function. If that were so, then the matrix T might be associated with the Euler factor at the real place.

We now remark on various numerology connected with n and $n + 1$ together that has appeared recently in several different number-theory contexts. In a paper of the first author with Harsh Mehta [9] we observed in a context of products of Farey fractions, but also potentially related to the Riemann hypothesis, functions with jumps that occur at a subset of these values k^2 and $k(k + 1)$, which arise in part as an artifact of Dirichlet’s hyperbola method. In another direction, work on splitting of polynomials of the first author with B. L. Weiss [11] led to the discovery by interpolation in a variable z of measures defined on each symmetric group S_n at $z = p$, a prime, to a signed measure at on the symmetric group S_n at the value $z = 1$, which combines symmetric group representations from S_n and S_{n-1} in an interesting way, and has an internal multiplicative structure respecting integer multiplication on n . This is being explored further in work in progress with Trevor Hyde.

Might there exist a family of finite-dimensional quantum integrable systems, with parameter n increasing to infinity, with a (possibly nonlinear) difference operator as a Hamiltonian, acting on a space of dimension higher than one, which can explain all these numerological coincidences?

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